数理解析研究所講究録
第 1596 巻 2008 年 69-73

Operator inequalities obtained from M. Uchiyama's recent results

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1 Remarks on Furuta inequality

In what follows, an operator means a bounded linear operator on a Hilbert space H. An operator T is positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and strictly positive (denoted by $T > 0$) if T is positive and invertible.

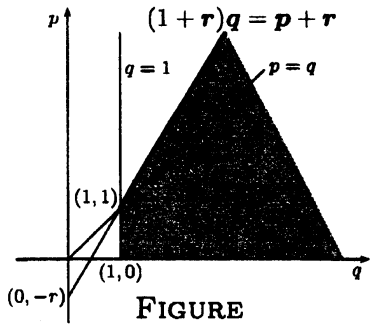
Theorem F (Furuta inequality [2]).
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $\quad (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $\quad (A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.



FIGURE

Löwner-Heinz theorem "$A \geq B \geq 0 \Longrightarrow A^\alpha \geq B^\alpha$ *for any* $\alpha \in [0,1]$" is the case $r = 0$ of Theorem F. Other proofs are given in [1][5] and also an elementary one-page proof in [3]. It is shown in [6] that the domain of p, q and r in Theorem F is the best possible for the inequalities (i) and (ii) to hold under the assumption $A \geq B$.

Remark 1. It was shown in [5] that $A \geq B \geq 0$ implies

$$B^{\frac{-r}{2}}(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}}B^{\frac{-r}{2}} \geq A^{\frac{p+r}{q}-r} \geq B^{\frac{p+r}{q}-r} \tag{1}$$

holds for $r \geq 0$, $p \geq 1$ and $q \geq 1$ with $(1+r)q \geq p+r$. The essential part of (1) is the first inequality, while the important condition $(1+r)q \geq p+r$ comes from the second.

Remark 2. Theorem F is based on the fact that

$$(B^{\frac{t}{2}}XB^{\frac{t}{2}})^{\frac{\delta}{\alpha}} \geq B^\delta \Longrightarrow (B^{\frac{t+u}{2}}XB^{\frac{t+u}{2}})^{\frac{\delta+u}{\alpha+u}} \geq B^{\delta+u} \tag{2}$$

holds for $B, X \geq 0$, $t \in \mathbb{R}$ and $0 \leq u \leq \delta \leq \alpha$. Theorem F can be proved by applying (2) repeatedly as follows: for $A, B \geq 0$ and $p \geq 1$,

$$
\begin{aligned}
A \geq B \Longleftrightarrow & (B^{\frac{0}{2}} A^p B^{\frac{0}{2}})^{\frac{1+0}{p+0}} \geq B^{1+0} \\
\Longrightarrow & (B^{\frac{0+u_1}{2}} A^p B^{\frac{0+u_1}{2}})^{\frac{1+0+u_1}{p+0+u_1}} \geq B^{1+0+u_1} \quad \text{for } u_1 \in [0,1] \text{ by } (2) \\
\Longrightarrow & (B^{\frac{1}{2}} A^p B^{\frac{1}{2}})^{\frac{1+1}{p+1}} \geq B^{1+1} \\
\Longrightarrow & (B^{\frac{1+u_2}{2}} A^p B^{\frac{1+u_2}{2}})^{\frac{1+1+u_2}{p+1+u_2}} \geq B^{1+1+u_2} \quad \text{for } u_2 \in [0,2] \text{ by } (2) \\
\Longrightarrow & (B^{\frac{3}{2}} A^p B^{\frac{3}{2}})^{\frac{1+3}{p+3}} \geq B^{1+3} \\
\Longrightarrow & (B^{\frac{3+u_3}{2}} A^p B^{\frac{3+u_3}{2}})^{\frac{1+3+u_3}{p+3+u_3}} \geq B^{1+3+u_3} \quad \text{for } u_3 \in [0,4] \text{ by } (2) \\
\Longrightarrow & \cdots .
\end{aligned}
$$

Proof of (2). The assumptions imply $(B^{\frac{t}{2}} X B^{\frac{t}{2}})^{\frac{u}{\alpha}} \geq B^u$ by Löwner-Heinz theorem, and there exists a contraction C such that $C^*(B^{\frac{t}{2}} X B^{\frac{t}{2}})^{\frac{u}{2\alpha}} = (B^{\frac{t}{2}} X B^{\frac{t}{2}})^{\frac{u}{2\alpha}} C = B^{\frac{u}{2}}$. Hence,

$$
\begin{aligned}
(B^{\frac{t+u}{2}} X B^{\frac{t+u}{2}})^{\frac{\delta+u}{\alpha+u}} &= \left(C^*(B^{\frac{t}{2}} X B^{\frac{t}{2}})^{\frac{\alpha+u}{\alpha}} C\right)^{\frac{\delta+u}{\alpha+u}} \\
&\geq C^*\left((B^{\frac{t}{2}} X B^{\frac{t}{2}})^{\frac{\alpha+u}{\alpha}}\right)^{\frac{\delta+u}{\alpha+u}} C \quad \text{by Hansen's inequality [4]} \\
&= B^{\frac{u}{2}} (B^{\frac{t}{2}} X B^{\frac{t}{2}})^{\frac{\delta}{\alpha}} B^{\frac{u}{2}} \\
&\geq B^{\delta+u} \quad \text{by the assumption.}
\end{aligned}
$$

□

In the *one-page* proof ([3]), the fact

$$
A \geq B \geq 0 \Longrightarrow (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geq B^{1+r} \text{ for } p \geq 1 \text{ and } r \in [0,1] \tag{3}
$$

is shown at first, and then (3) is used doubly and nestedly as

$$
A \geq B \geq 0 \Longrightarrow A_1 \geq B_1 \Longrightarrow (B_1^{\frac{r_1}{2}} A_1^{p_1} B_1^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}} \geq B_1^{1+r_1}
$$

where $A_1 = (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1+r}{p+r}}$, $B_1 = B^{1+r}$, $p_1 = \frac{p+r}{1+r}$ and $r_1 = 1$. We note that the value of p_1 is chosen in order that $h(t) = t^{p_1}$ becomes the inverse function of $\varphi(t) = t^{\frac{1+r}{p+r}}$. It might be remarkable that in the proof of (2), we use neither such an implication proposition with the hypothesis $A \geq B$ as (3) nor such an inverse function as $h(t)$.

2 Uchiyama's results and their generalizations

Let $\mathbb{P}_+[a,b)$ be the set of all non-negative operator monotone functions defined on $[a,b)$, and $\mathbb{P}_+^{-1}[a,b)$ the set of increasing functions h defined on $[a,b)$ such that $h([a,b)) = [0,\infty)$ and its inverse h^{-1} is operator monotone on $[0,\infty)$. Uchiyama [7] introduces a new concept of majorization, and shows a quite interesting result named "Product theorem."

Definition ([7]). Let h be a non-decreasing function on I and k an increasing function on J. Then h is said to be majorized by k, in symbols $h \preceq k$, if $J \subseteq I$ and the composite $h \circ k^{-1}$ is operator monotone on $k(J)$.

Product theorem ([7]). *Suppose $-\infty < a < b \leq \infty$. Then*

$$\mathbb{P}_+[a,b) \cdot \mathbb{P}_+^{-1}[a,b) \subseteq \mathbb{P}_+^{-1}[a,b), \quad \mathbb{P}_+^{-1}[a,b) \cdot \mathbb{P}_+^{-1}[a,b) \subseteq \mathbb{P}_+^{-1}[a,b).$$

Further, let $h_i \in \mathbb{P}_+^{-1}[a,b)$ for $1 \leq i \leq m$, and let g_j be a finite product of functions in $\mathbb{P}_+[a,b)$ for $1 \leq j \leq n$. Then for $\psi_i, \phi_j \in \mathbb{P}_+[0,\infty)$

$$\prod_{i=1}^m h_i(t) \prod_{j=1}^n g_j(t) \in \mathbb{P}_+^{-1}[a,b), \quad \prod_{i=1}^m \psi_i(h_i(t)) \prod_{j=1}^n \phi_j(g_j(t)) \preceq \prod_{i=1}^m h_i(t) \prod_{j=1}^n g_j(t).$$

Furthermore, he applies Product theorem to obtain generalizations of Theorem F.

Proposition A ([7]). *Let $h \in \mathbb{P}_+^{-1}[0,\infty)$, and let $\tilde{h}$ be a non-negative non-decreasing function on $[0,\infty)$ such that $\tilde{h} \preceq h$. Let g be a finite product of functions in $\mathbb{P}_+[0,\infty)$. Then for the function φ defined by $\varphi(h(t)g(t)) = \tilde{h}(t)g(t)$*

$$A \geq B \geq 0 \Longrightarrow \begin{cases} \varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \geq g(B)^{\frac{1}{2}}\tilde{h}(A)g(B)^{\frac{1}{2}}, \\ \varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \leq g(A)^{\frac{1}{2}}\tilde{h}(B)g(A)^{\frac{1}{2}}. \end{cases}$$

Theorem B ([7]). *Let $h \in \mathbb{P}_+^{-1}[0,\infty)$, and let $\tilde{h}$ be a non-negative non-decreasing function on $[0,\infty)$ such that $\tilde{h} \preceq h$. Let g_n be a finite product of functions in $\mathbb{P}_+[0,\infty)$ for each n, and let the sequence $\{g_n\}$ converge pointwise to g. Suppose $g \neq 0$ and $g(0+) = g(0)$. Then for the function φ defined by $\varphi(h(t)g(t)) = \tilde{h}(t)g(t)$*

$$A \geq B \geq 0 \Longrightarrow \begin{cases} \varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \geq g(B)^{\frac{1}{2}}\tilde{h}(A)g(B)^{\frac{1}{2}}, \\ \varphi(g(A)^{\frac{1}{2}}h(B)g(A)^{\frac{1}{2}}) \leq g(A)^{\frac{1}{2}}\tilde{h}(B)g(A)^{\frac{1}{2}}. \end{cases}$$

We obtain extensions of Proposition A and Theorem B by weakening their hypotheses from $A \geq B$ to inequalities implied by it. We note that these results are slightly improved versions of those in [8] from the viewpoint of the remarks in the previous section.

Proposition 1. *Let f_i be non-negative non-decreasing functions on $[0,\infty)$ and $g_j(t) = \prod_{i=1}^j f_i(t)$. Let h, $\hat{h}$ and $\tilde{h}$ be non-negative non-decreasing functions on $[0,\infty)$ such that $f_n(t) \preceq \hat{h}(t)g_{n-1}(t)$, $\tilde{h} \preceq h$ and $h(0)g_{n-1}(0) = 0$. Then for the functions ψ_j and φ_j defined by $\psi_j(h(t)g_j(t)) = \hat{h}(t)g_j(t)$ and $\varphi_j(h(t)g_j(t)) = \tilde{h}(t)g_j(t)$, if $A, B \geq 0$ satisfy*

$$\psi_{n-1}(g_{n-1}(B)^{\frac{1}{2}}h(A)g_{n-1}(B)^{\frac{1}{2}}) \geq \hat{h}(B)g_{n-1}(B),$$

then

$$\varphi_n(g_n(B)^{\frac{1}{2}}h(A)g_n(B)^{\frac{1}{2}}) \geq f_n(B)^{\frac{1}{2}}\varphi_{n-1}(g_{n-1}(B)^{\frac{1}{2}}h(A)g_{n-1}(B)^{\frac{1}{2}})f_n(B)^{\frac{1}{2}}$$

holds. Furthermore,

$$\psi_n(g_n(B)^{\frac{1}{2}}h(A)g_n(B)^{\frac{1}{2}}) \geq \hat{h}(B)g_n(B)$$

holds if $\hat{h} \preceq h$.

Theorem 2. *Let $\hat{h} \in \mathbb{P}_+^{-1}[0,\infty)$, and let h and $\tilde{h}$ be non-negative non-decreasing functions on $[0,\infty)$ such that $\tilde{h} \preceq h$ and $\hat{h} \preceq h$. Let g be a finite product of functions in $\mathbb{P}_+[0,\infty) \cup \mathbb{P}_+^{-1}[0,\infty)$ and γ_n a finite product of functions in $\mathbb{P}_+[0,\infty)$ for each n, and let the sequence $\{g(t)\gamma_n(t)\}$ converge pointwise to $\bar{g}(t)$. Suppose $\bar{g} \neq 0$ and $\bar{g}(0+) = \bar{g}(0)$. Then for the functions ψ, $\bar{\psi}$, φ and $\bar{\varphi}$ defined by $\psi(h(t)g(t)) = \hat{h}(t)g(t)$, $\bar{\psi}(h(t)\bar{g}(t)) = \hat{h}(t)\bar{g}(t)$, $\varphi(h(t)g(t)) = \tilde{h}(t)g(t)$ and $\bar{\varphi}(h(t)\bar{g}(t)) = \tilde{h}(t)\bar{g}(t)$, if $A, B \geq 0$ satisfy*

$$\psi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}}) \geq \hat{h}(B)g(B),$$

then

$$g(B)^{\frac{1}{2}}\bar{\varphi}(\bar{g}(B)^{\frac{1}{2}}h(A)\bar{g}(B)^{\frac{1}{2}})g(B)^{\frac{1}{2}} \geq \bar{g}(B)^{\frac{1}{2}}\varphi(g(B)^{\frac{1}{2}}h(A)g(B)^{\frac{1}{2}})\bar{g}(B)^{\frac{1}{2}}$$

and

$$\bar{\psi}(\bar{g}(B)^{\frac{1}{2}}h(A)\bar{g}(B)^{\frac{1}{2}}) \geq \hat{h}(B)\bar{g}(B)$$

hold.

Proof of Proposition 1 $\Longrightarrow$ Proposition A. Put $\hat{h}(t) = t$ and $f_1(t) = g_1(t) = 1$, then

$$\psi_1(g_1(B)^{\frac{1}{2}}h(A)g_1(B)^{\frac{1}{2}}) = \psi_1(h(A)g_1(A)^{\frac{1}{2}}) = \hat{h}(A)g_1(A) = A \geq B = h(B)g_1(B).$$

By applying Proposition 1, we have

$$\begin{aligned}\psi_1(g_1(B)^{\frac{1}{2}}h(A)g_1(B)^{\frac{1}{2}}) \geq h(B)g_1(B) &\Longrightarrow \psi_2(g_2(B)^{\frac{1}{2}}h(A)g_2(B)^{\frac{1}{2}}) \geq h(B)g_2(B)\\ &\Longrightarrow \psi_3(g_3(B)^{\frac{1}{2}}h(A)g_3(B)^{\frac{1}{2}}) \geq h(B)g_3(B)\\ &\Longrightarrow \cdots\\ &\Longrightarrow \psi_{n-1}(g_{n-1}(B)^{\frac{1}{2}}h(A)g_{n-1}(B)^{\frac{1}{2}}) \geq h(B)g_{n-1}(B)\end{aligned}$$

since $\hat{h}(t) = t \preceq h(t)$, and

$$\begin{aligned}&\psi_k(g_k(B)^{\frac{1}{2}}h(A)g_k(B)^{\frac{1}{2}}) \geq h(B)g_k(B)\\ &\quad\Longrightarrow \varphi_{k+1}(g_{k+1}(B)^{\frac{1}{2}}h(A)g_{k+1}(B)^{\frac{1}{2}}) \geq f_{k+1}(B)^{\frac{1}{2}}\varphi_k(g_k(B)^{\frac{1}{2}}h(A)g_k(B)^{\frac{1}{2}})f_{k+1}(B)^{\frac{1}{2}}\end{aligned}$$

for $k = 1, 2, \ldots, n-1$. Therefore

$$\begin{aligned}\varphi_n(g_n(B)^{\frac{1}{2}}h(A)g_n(B)^{\frac{1}{2}}) &\geq f_n(B)^{\frac{1}{2}}\varphi_{n-1}(g_{n-1}(B)^{\frac{1}{2}}h(A)g_{n-1}(B)^{\frac{1}{2}})f_n(B)^{\frac{1}{2}}\\ &\geq f_n(B)^{\frac{1}{2}}f_{n-1}(B)^{\frac{1}{2}}\varphi_{n-2}(g_{n-2}(B)^{\frac{1}{2}}h(A)g_{n-2}(B)^{\frac{1}{2}})f_{n-1}(B)^{\frac{1}{2}}f_n(B)^{\frac{1}{2}}\\ &\geq \cdots\\ &\geq f_n(B)^{\frac{1}{2}}\cdots f_2(B)^{\frac{1}{2}}\varphi_1(g_1(B)^{\frac{1}{2}}h(A)g_1(B)^{\frac{1}{2}})f_2(B)^{\frac{1}{2}}\cdots f_n(B)^{\frac{1}{2}}\\ &= g_n(B)^{\frac{1}{2}}\tilde{h}(A)g_n(B)^{\frac{1}{2}}.\end{aligned}$$

□

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